TOPOLOGY II - SEMESTRAL EXAM, 2024

Time : 3 hours | Max. Marks : 60

Answer all questions. You may use results proved in class after correctly quoting them. Any other claim must be accompanied by a proof.

(1) (a) Decide whether the map $p : [0, \infty) \longrightarrow S^1$, $p(t) = e^{2\pi i t}$ is a covering map. [4]
(b) Show that the free group $\mathbb{Z} * \mathbb{Z}$ of rank two has a normal subgroup of index 3. [4]
(c) Construct a connected covering of $S^1 \vee S^1$ corresponding to the subgroup $<< a^2, b^2 >>$ of $\pi_1(S^1 \vee S^1)$. Here a, b have the usual meaning. [10]

(2) (a) Given a pair (X,A), prove that $H_0(X, A; \mathbb{Z}) = 0$ if and only if A intersects every path component of X. [6]
(b) Define the degree of a map $f : S^n \longrightarrow S^n$. Let $f : S^n \longrightarrow S^n$ be a map that satisfies $f(x) = f(-x)$ for all $x \in S^n$. Compute the degree of f. [8]
(c) Define the notion of a CW-complex. Prove that $\mathbb{R}P^n$ is a CW-complex with one i-cell for each i, $0 \leq i \leq n$. [2+5]

(3) (a) Define a good pair. Show that if (X, A) is a good pair, then the homomorphism

$$p_* : H_q(X, A; \mathbb{Z}) \longrightarrow \widetilde{H}_q(X/A; \mathbb{Z})$$

induced by the quotient map $p : X \longrightarrow X/A$ is an isomorphism for all $q \geq 0$. [8]
(b) Construct a map $g : S^1 \times S^1 \longrightarrow S^2$ that is not null homotopic. [6]
(c) Show that the quotient map $p : \mathbb{R}P^3 \longrightarrow RP^3/RP^2$ is not null homotopic. [7]